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Rapid Note

Damage-spreading in the Bak-Sneppen model without noise

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Abstract. We study the behavior under perturbations in the, recently introduced, Bak-Sneppen model with deterministic updating. We focus our attention on the damage-spreading features and show that the value of the growth exponent for the distance, $\alpha = 0.32$, coincides with that of the random updating Bak-Sneppen model. Moreover, we generalize this analysis by considering a broader set of initial perturbations for which the value of α is preserved.

PACS. 05.20.-y Statistical mechanics – 05.45.+b Theory and models of chaotic systems – 05.70.Ln Nonequilibrium thermodynamics, irreversible processes

A great deal of evidence has been put forward in recent years for the appearance of criticality in nature: a wide variety of phenomena, from biological evolution [1] to earthquakes [2], from surface growth [3] to fluid displacement in porous media [4], exhibit scale invariance in both space and time. Scale invariance means that the correlation length in these systems is infinite and consequently, a small (local) perturbation can produce a global (maybe even drastic) effect. This possibility leads naturally to the study of the sensitivity to perturbations in critical systems.

To study the propagation of local perturbations (*damage spreading*) one can borrow a technique from dynamical systems theory. Let us consider, for instance, two copies of the same dynamical system, with slightly different initial conditions. By following the dynamics of both copies and studying the evolution in time of the "distance" $D(t)$ between them, it is possible to quantify the effect of the initial perturbation. Indeed, assuming that the distance $D(t)$ grows exponentially, and defining the Lyapunov exponent λ *via*

$$D(t) = D_0 \exp(\lambda\, t), \tag{1}$$

three different behaviors can be distinguished, corresponding to λ being either positive, negative or zero. The case $\lambda > 0$ corresponds to the so-called *chaotic* systems, where the extremely high sensibility to initial conditions leads to exponentially diverging trajectories. The case $\lambda < 0$, instead, characterizes those systems in which the dynamics has an attractor and any initial perturbation is "washed out" with exponential rapidity.

The boundary case, $\lambda = 0$, admits, in turn, a whole class of functions $D(t)$, namely

$$D(t) \sim t^{\alpha} \tag{2}$$

where α is some exponent, characteristic of the system. In particular, $\alpha > 0$ corresponds to weak sensitivity to initial conditions while $\alpha < 0$ corresponds to weak insensitivity to initial conditions (as an example, the reader is referred to Refs. [5], where this analysis is performed for the logistic map at its critical point). In [5], the behavior described by equation (2) has been related to the non-extensivity of the entropy proposed in [7]. Indeed, the exponent α can be shown to be $\alpha = 1/1 - q$ where q is the extensivity parameter ($q = 1$ corresponding to the extensive case).

Recently [6], this analysis was performed on the Bak-Sneppen (BS) model [1]. Originally proposed to describe ecological evolution, this model has been paid a great deal of attention due to its simplicity and the fact that it exhibits self-organized criticality [8]. Its critical properties allow us to describe its behavior under perturbations *via* equation (2), with

$$\alpha = 0.32\,. \tag{3}$$

In a recent paper [9], the results presented in [6] were explained by relating the BS model to a simpler model. The purpose of this paper is twofold. First of all, we analyze, with the methods introduced in [6,9], the recently introduced deterministic BS model [10], to show that the exponent α is constant (as it should be) within the BS universality class. In the second part of the paper, we extend the analysis of [6,9] to show that the nature of the initial perturbation is actually irrelevant. Indeed, the same

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growth exponent for the distance is obtained for a whole class of perturbations.

In its simplest version, the BS model describes an ecosystem as a collection of N species on a one dimensional lattice. To each species corresponds a fitness described by a number f between 0 and 1. For simplicity, one considers periodic boundary conditions. The initial state of the system is defined by assigning to each site j a random fitness f_0^j chosen from a uniform distribution. The dynamics proceeds in three basic steps.

1. Find the site with the absolute minimum fitness on the lattice (the active site) and its two nearest neighbors.
2. Update the values of their fitnesses by assigning to them new random numbers from a uniform distribution.
3. Return to step 1.

After an initial transient that will be of no interest to us here, a non-trivial critical state is reached. This critical state, characterized by its statistical properties, can be understood as the *fluctuating balance* between two competing "forces". Indeed, while the random assignation of the values, together with the coupling, acts as an entropic disorder, the choice of the minimum acts as an ordering force. As a result of this competition, at the stationary state the majority of the f^j have values above a certain threshold f_c. Only a few will be below f_c, namely those belonging to the running avalanche (see [1,10,11] for a detailed discussion).

Through the use of different maps (chaotic as well as non-chaotic), it was shown [10] that the random updating is not a necessary requirement to have SOC. Moreover, as long as the updating rule is chaotic the system does not change the universality class, *i.e.* all the exponents are the same as in the case of random updating. This means that the system is able to self-organize at a higher level: it takes into account the temporal correlation (or the average time spent in every site) by increasing the threshold, so as to have the same statistical properties [10]. As a consequence, all equations and relations derived for the original BS model are still valid for all the cases with chaotic updating. The stationary distribution of the fitnesses, on the other hand, follows a different pattern. Indeed, the position of the threshold as well as the exact shape of the stationary distribution depends on the actual form of the updating rule.

As mentioned above, to study the behavior under perturbations one produces two identical copies B_1 and B_2 of the system in the critical state, and finds the minimum (the active site). Then, a slight perturbation is introduced in B_2 (as explained later on) and the evolution of the Hamming distance

$$D(t) \;=\; \frac{1}{N}\sum_{j=1}^{N} |f_j^1 - f_j^2| \qquad (4)$$

is followed in time. Since this quantity has strong fluctuations, we will consider the average $\langle D(t)\rangle$, over realizations. To calculate the value of $\langle D(1)\rangle$ one needs to specify the nature of the perturbation. One option is to swap the position of the minimum in B_2 with any site taken at random. With this prescription, at $t = 1$, the average (initial) distance $\langle D(1)\rangle$ can be obtained from equation (4),

$$\langle D(1)\rangle \;=\; \frac{2}{N}\int_0^1 df^1 df^2 \eta_1(f^1)\eta_2(f^2)|f^1 - f^2|, \qquad (5)$$

where η_i is the distribution function (at $t = 1$) for the variable $f^i \in B_i$. Applying a similar procedure, one can study the growth of the distance for $1 \ll t \ll N$. In fact, it is enough to observe that

$$\langle D(t)\rangle \;=\; \langle D(1)\rangle\, \bar{n}_{cov}(t)\,, \qquad (6)$$

where $\bar{n}_{cov}(t)$ is the averaged number of different sites covered in the two copies of the system at time t. As explained in [9], $\bar{n}_{cov}(t)$ may depend on the internal correlation of the system and on the correlations between the two copies. In the 1D Bak-Sneppen model, the growth rate cannot give an exponent $\alpha > 1$ and stops at a certain time $\tau \sim N^z$ at which a crossover to a saturation regime appears. Clearly, this is due to the fact that after τ time-steps each site of the lattice has been covered at least once. For $t \gg \tau$, almost all the lattice sites have been covered and the two strings are made of the same random numbers placed in different position along the lattice. Thus, the distance reaches a plateau, independent on the size N of the system, given by

$$\begin{aligned} D_{asym} &= \langle D(t \to \infty)\rangle \\ &= \int_0^1 df^1 df^2 \rho_1(f^1)\rho_2(f^2)\,|f^1 - f^2|\,, \end{aligned} \qquad (7)$$

where ρ_i is the normalized distribution function (at $t = \infty$) for the variable $f^i \in B_i$ [13].

In the model with random updating, the initial distance can be computed using equation (5) and the fact that both distributions are (roughly) given by

$$\eta_1(f) = \left(3 - \frac{9}{2}\,f\right)\Theta\left(\frac{2}{3} - f\right) \qquad (8)$$

$$\eta_2(f) = 3\,\Theta\left(f - \frac{2}{3}\right), \qquad (9)$$

for the distribution of the active sites and of the sites above threshold respectively (the threshold has been put equal to $2/3$ in first approximation and Θ is the step function). Inserting equations (8, 9) in equation (5) we obtain $\langle D(1)\rangle \sim 1.2/N$. The saturation value is instead obtained from equation (7) with $\rho_1 = \rho_2 = \eta_2$, where η_2 comes from equation (9), and reads $D_{asym} = \langle D(t \to \infty)\rangle \sim 0.11$. Therefore, the saturation value does not depend on the size of the system while the initial distance does. Thus, the normalized distance $\langle D(t)\rangle/\langle D(1)\rangle$ reaches a plateau that must scale with N, as confirmed by numerical simulations [9,12]. When one considers the chaotic updating, equations (5, 7) are still valid. The only difference is that one needs to find the stationary distributions that correspond to the actual map. For instance, for the tent map

Table 1. Values of α for the different updating rules and perturbation prescriptions. The system size is $N = 2 \times 10^3$. All the values are in good agreement with the value computed in [10], by using the flipping perturbation only.

	flipping	random perturbation
random updating	$\alpha = 0.33(2)$	$\alpha = 0.33(2)$
logistic updating	$\alpha = 0.31(2)$	$\alpha = 0.29(2)$

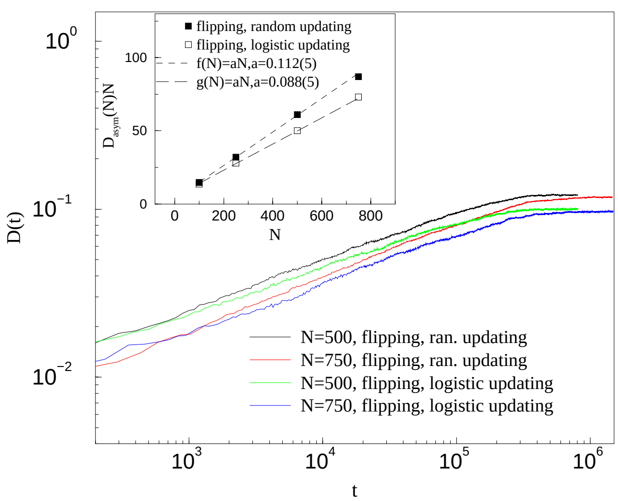

Fig. 1. The $\log_{10}$-$\log_{10}$ plot of $D(t)$ for the BS model with flipping, with random updating and deterministic updating with logistic map, for different system sizes. The value of the plateau actually depends on the updating rule, but it does not depend on the system size. Inset: a plot of the plateau of $ND(t)$ *versus* N fits very well with a linear scaling, showing that the plateau of the Hamming distance $D(t)$ is size independent.

as well as for the Bernoulli map the distribution coincides with that of the random updating (except for the value of the threshold in the Bernoulli case) [10]. As an example, we substitute the random updating with the logistic map

$$f_i(t+1) = b f_i(t)(1 - f_i(t)), \qquad (10)$$

where i runs over the minimum and its nearest neighbors and b is a parameter (that we set to the value 4 which is at the threshold of the chaotic phase). Inserting the estimated distributions obtained in [10] one obtains as first approximation $D_{asym} \sim 0.087$ and the same exponent α as in the random updating (see Tab. 1). In Figure 1 one can see the evolution of the distance for the case of a Bak-Sneppen model with random and logistic updating, using the perturbation defined in [6] (flipping of the minimum fitness with another fitness chosen at random). Simulations with other chaotic maps give the same exponents too: this reflects the observed fact that chaotic maps do not change the universality class of the model [10]. In the inset we show the scaling of $ND_{asym}(N)$ *versus* N for both random and logistic updating. A linear fit gives as estimation of the plateau $D_{asym} = 0.112(5)$ for random updating and $D_{asym} = 0.088(5)$ for logistic updating, in good agreement with our analytic estimate.

One point remains, however, that needs to be studied. The definition of the initial perturbation in the replica in [6] is too restrictive. In fact, by considering as initial perturbation

$$\tilde{f}_i = f_i + \epsilon g_i, \qquad (11)$$

where ϵ is a small positive number, we can take several choices for g_i without altering the exponent α. We considered four different implementations of g_i:

(a) $g_i = \psi(t)$ where $\psi(t)$ is a random number between 0 and 1;
(b) $g_i = \psi(t)$ where $\psi(t)$ is a random number between $-1/2$ and $1/2$;
(c) $g_i = \psi(t)$ for i corresponding to the minimum and zero otherwise;
(d) $g_i = constant = 1$ for all sites.

This kind of perturbation allows us to tune the initial mean distance $D(t)$, to any arbitrarily small value depending on ϵ in equation (11) (contrary on the flipping introduced in [6], which gives a fixed, size dependent, mean initial distance). In case (d), for example, (which is the case we will use in the analysis below) the initial distance is $D(1) = \epsilon$, independent on N, and the plateau will be independent on N too. This characteristic of the global perturbation is useful to explore some properties of the model with deterministic updating and is more in line with the standard techniques of damage spreading problems.

We have performed the analysis with the prescription (11) also in the case in which the model has a deterministic microscopic rule, like an updating with the logistic map. In this case, the numbers in each replica will not be the same since the map will be applied on different numbers. In Figure 2 we show the behavior of $D(t)$ for random and logistic updating, where we used perturbation (11) with the implementation (d). The exponent does not change: in both cases a value of α around the value 0.32 found in [6] is obtained (see Tab. 1). The plateau of $ND(t)$ (we computed $ND(t)$ to reduce statistical fluctuations) for the case of global perturbation, scales linearly with the system size N, as shown in the inset of Figure 2, with a slope $a = D_{asym} = 0.114(5)$ and $a = D_{asym} = 0.089(5)$ respectively for random updating and logistic updating. The good agreement with the rough analytic estimate obtained above and with numerical results for the flipping show that the value of the plateau, too, does not depend on the kind of perturbation applied. Consequently, the distance $D(t)$ has a plateau D_{asym} independent on the size, as it happens with the flipping. These results do not change if we use the implementations (a), (b), (c) of the perturbation (11) [12].

The case of the logistic map with perturbation (11) is particularly interesting from a different point of view. Since the map is chaotic, one could expect that on small scale distances ($D(t) \ll 1/N$) the chaoticity of the maps dominates and the distance of the two copies grows exponentially. For bigger distances instead, the dynamics is dominated by the damage spreading (and thus the critical properties of the BS model) and the distance

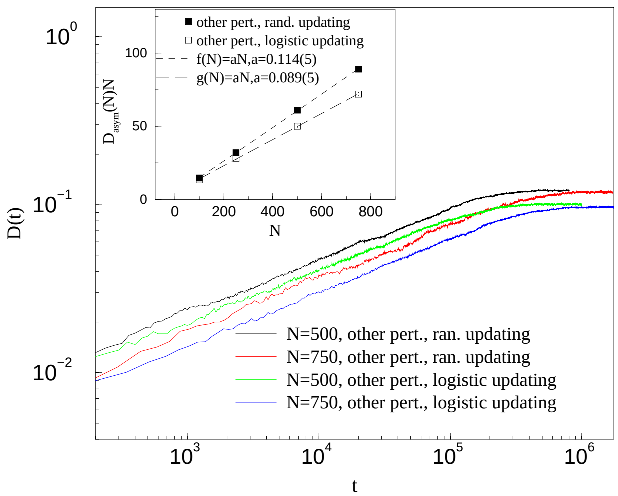

Fig. 2. The $\log_{10}$-$\log_{10}$ plot of $D(t)$ for the BS model with the perturbation (11), implementation (d), with random updating and deterministic updating with logistic map, for different system sizes. The value of the plateau actually depends on the updating rule, as in the flipping case. Inset: a plot of the plateau of $ND(t)$ *versus* N gives the same findings of the case of flipping.

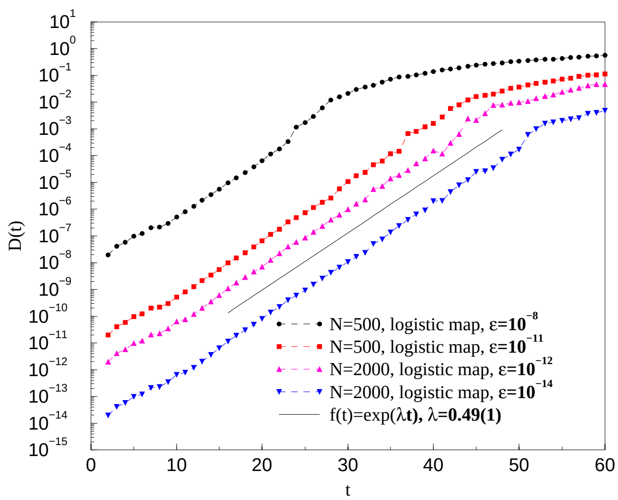

Fig. 3. The *linear*-$\log_{10}$ plot of $D(t)$ for the BS model with perturbation (11), implementation (d), and updating with a logistic map with parameter $b = 4$, for different system sizes and mean initial distances ϵ.

grows as a power law. This is exactly what we find numerically. In Figure 3 we show some numerical simulations of the BS model with deterministic updating with the logistic map (with parameter value 4) and random perturbation, for different system sizes and initial distances.

The distance $D(t)$ has an initial exponentially growing phase with a Lyapunov exponent $\lambda = 0.49(1)$. This Lyapunov exponent is smaller than the Lyapunov exponent of a single logistic map with $b = 4$, which actually is $\log 2 = 0.693147...$ Indeed, the microscopic dynamical rule of the system changes the values of the minimum fitness and its nearest neighbors, and the Lyapunov exponent we measure arises from the interaction between the three maps applied to the three different numbers.

Summarizing, the behavior under perturbation of the BS model with deterministic updating resembles very much that of the original BS model. The final value of the plateau of the normalized distance depends on the map. It is only in the short time limit, for times of the order of the microscopic time, that any difference can be seen. In fact in this regime, an exponential growth of the distance is observed.

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